Eigenfunctions of Systems of Linear Ordinary Differential Equations with Separated Boundary Conditions Using Riccati Transformations

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The Riccati transformation has been used previously to calculate eigenvalues of systems of ordinary differential equations with separated boundary conditions. This note describes how eigenfunctions of such differential systems may also be obtained using Riccati methods. The techniques are illustrated by numerical examples which involve evaluation of real and complex eigenfunctions.

1. INTRODUCTION

Scott [1] introduced the Riccati transformation method for the computation of eigenvalues of a system of linear ordinary differential equations of the form

$$du/dz = A(z, \sigma)\,\mathbf{u} + B(z, \sigma)\,\mathbf{v}$$
$$-(d\mathbf{v}/dz) = C(z, \sigma)\,\mathbf{u} + D(z, \sigma)\,\mathbf{v}, \tag{1}$$

under the linear separated boundary conditions

$$\mathbf{u}(0) = \mathbf{0}, \qquad \mathbf{u}(x) = \mathbf{0} \quad \text{or} \quad \mathbf{v}(x) = \mathbf{0}. \tag{2}$$

Here $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$ and $A(z, \sigma)$, $B(z, \sigma)$, $C(z, \sigma)$, and $D(z, \sigma)$ are $n \times n$ real matrices which depend on the independent variable z and on some scalar parameter σ. This work was extended by Sloan and Wilks [2] to deal with the general linear separated boundary conditions

$$\alpha_1\mathbf{u}(0) + \beta_1\mathbf{v}(0) = \mathbf{0} \qquad \alpha_2\mathbf{u}(x) + \beta_2\mathbf{v}(x) = \mathbf{0}, \tag{3}$$

where the real matrices $[\alpha_1\beta_1]$ and $[\alpha_2\beta_2]$ have dimensions $n \times 2n$ and rank n.

The objective of this paper is to show that the Riccati method may be used in a very straightforward manner to compute the eigenfunctions $\mathbf{u}$ and $\mathbf{v}$ which satisfy (1) and (3). Scott [3] has considered the problem of evaluating eigenfunctions of (1) using Riccati methods in the special case $n = 1$. Scott's method involves the integration of a scalar Riccati equation and an associated linear equation from $z = 0$ in the direction of increasing z; the approach has certain aspects in common with one of the methods considered here for a system with $n \geqslant 1$. Section 2 of this paper discusses the role played by the eigenfunctions in the Riccati approach and sets up

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suitable equations for the evaluation of the eigenfunctions. Two illustrative examples of the theory are given in Section 3: The first is a differential system with real eigenfunctions and the second is the Orr–Sommerfeld equation which involves complex coefficients and complex eigenfunctions. The final section notes some of the difficulties involved in the numerical solution of differential eigenproblems and indicates possible advantages of the Riccati approach. Further areas of research are mentioned.

The results contained in Refs. [1, 2] are basic to subsequent discussions and it is convenient to have at hand a summary of relevant material from these references. In the accompanying paper Wilks and Bramley [4] consider the application of Riccati methods to an odd order system over a semi-infinite interval. The forementioned summary is contained in Section 2 of [4] and, to avoid unnecessary repetition, the summary will be regarded as part of the introduction to this paper. We need only repeat the key equations. Any reference to an equation in [4] will have the letters WB attached to the equation number; Eq. (12), for example, will be referred to as (WB12).

The Riccati method involves a transformation to new dependent variables $U(z) = \alpha_1 u(z) + \beta_1 v(z)$ and $V(z) = \gamma_1 u(z) + \delta_1 v(z)$, where the $2n \times 2n$ matrix $M = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix}$ is nonsingular [4]. The transformed boundary conditions are (WB10) at $z = 0$ and

$$\alpha U(x) + \beta V(x) = 0 \tag{4}$$

at $z = x$, where $[\alpha \ \beta]$ is an $n \times 2n$ matrix of rank n. If $U(z)$ and $V(z)$ are related by

$$U(z) = E(z) V(z) \tag{5}$$

then $E(z)$ will satisfy the matrix Riccati equation (WB12) and the condition at $z = 0$ indicates that this equation may be integrated from $E(0) = 0$. The approach adopted in [4] is to transform from $E(z)$ to a new matrix $R(z)$ after the integration has commenced, with $R(z)$ chosen to have zero determinant at characteristic lengths. However, Eqs (4) and (5) show that characteristic lengths may be located using the equation

$$\det[\alpha E(x) + \beta] = 0 \tag{6}$$

provided $\alpha \neq 0$, which is the case if γ_1 and δ_1 are chosen suitably. In the course of the integration from $z = 0$ to $z = x$, any singularities of $\det E(z)$ may be traversed by transforming to a Riccati system in the matrix $E^{-1}(z)$

2. COMPUTATION OF EIGENFUNCTION

2.1. *Solution Vectors and Riccati Matrices*

Prior to proposing methods for evaluating eigenfunctions it is of some importance to consider how the existence of eigenfunctions is related to the behavior of the Riccati matrices. Suppose the problem defined by (1) and (3) is written as

$$d\mathbf{y}/dz = L\mathbf{y}, \tag{7}$$
$$\hat{B}\mathbf{y}(0) = \mathbf{0}, \tag{8a}$$
$$\hat{C}\mathbf{y}(x) = \mathbf{0}, \tag{8b}$$

where $\mathbf{y} = [\begin{smallmatrix}\mathbf{u}\\\mathbf{v}\end{smallmatrix}]$, $\hat{B} = [\alpha_1 \; \beta_1]$, $\hat{C} = [\alpha_2 \; \beta_2]$, and L is the obvious $2n \times 2n$ coefficient matrix. If $\Sigma(z)$ denotes the space of solutions of (7) which satisfy the initial condition (8a) then at any station z this space will be a vector space of dimension n. Under the transformation (WB9) any element $\mathbf{y}(z)$ maps onto $\mathbf{Y}(z) = [\begin{smallmatrix}\mathbf{U}\\\mathbf{V}\end{smallmatrix}]$ and the transformed boundary conditions are

$$[I \; 0] \, \mathbf{Y}(0) = \mathbf{0}, \tag{9a}$$

$$[\alpha \; \beta] \, \mathbf{Y}(x) = \mathbf{0}. \tag{9b}$$

The use of transformation (5) effectively assumes that any element $\mathbf{Y}(z)$ of $\Sigma(z)$ may be represented as a linear combination of the columns of a matrix

$$\begin{bmatrix} E(z) \\ I \end{bmatrix} V(z), \tag{10}$$

where the columns of the $n \times n$ matrix $V(z)$ are linearly independent and they may be regarded as a basis for the solutions $\mathbf{V}(z)$. With $\mathbf{Y}(z)$ represented in terms of the basis (10) we see that if in the course of the integration a point z is reached at which there is a $\mathbf{Y}(z)$ in $\Sigma(z)$ and in the null space of $[I \; 0]$, denoted by $N([I \; 0])$, then $\det[E(z)]$ is necessarily zero. The terminating condition (9b) will be satisfied at any point $z = x$ where there is a vector $\mathbf{Y}(z)$ in $\Sigma(z)$ and in $N([\alpha \; \beta])$. With $\mathbf{Y}(z)$ represented by the basis (10) a necessary and sufficient condition for the existence of such a common vector is that $\det[\alpha E(z) + \beta] = 0$ at $z = x$. The vector $\mathbf{Y}(x)$ which is in $\Sigma(x)$ and $N([\alpha \; \beta])$ is an eigenfunction of the given differential system evaluated at $z = x$. The problem of finding an eigenfunction for all z in $0 \leqslant z \leqslant x$ is that of finding the element of $\Sigma(z)$ which matches $\mathbf{Y}(x)$ at $z = x$. The vector $\mathbf{Y}(x)$ is given by $[\begin{smallmatrix}\mathbf{U}(x)\\\mathbf{V}(x)\end{smallmatrix}]$, where $\mathbf{V}(x)$ is a suitably normalized solution of $[\alpha \; E(x) + \beta] \, \mathbf{V}(x) = \mathbf{0}$, and $\mathbf{U}(x) = E(x) \, \mathbf{V}(x)$. If the eigenfunction $\mathbf{Y}(z)$ has been obtained we may obtain the solution $\mathbf{y}(z)$ of Eqs. (7) and (8) by means of the transformation $\mathbf{y}(z) = M^{-1} \, \mathbf{Y}(z)$.

Note that less general boundary conditions are dealt with in the obvious manner. For example, to solve Eq. (1) with boundary conditions $\mathbf{u}(0) = \mathbf{0}$, $\mathbf{u}(x) = \mathbf{0}$ we use the matrix R defined by (WB3). At $z = x$ an eigenfunction is a vector $\mathbf{y}(x)$ which is in $\Sigma(x)$ and $N([I \; 0])$ so that $\mathbf{y}(x)$ is given by $[\begin{smallmatrix}\mathbf{0}\\\mathbf{v}(x)\end{smallmatrix}]$, where $\mathbf{v}(x)$ is a suitable solution of $R(x) \, \mathbf{v}(x) = \mathbf{0}$. If the terminating condition is $\mathbf{v}(x) = \mathbf{0}$ then $\mathbf{y}(x) = [\begin{smallmatrix}\mathbf{u}(x)\\\mathbf{0}\end{smallmatrix}]$, where $S(x) \, \mathbf{u}(x) = \mathbf{0}$. Note also that if a problem is being solved for a specified value of x we assume that the parameter σ in (1) has been adjusted until there is a vector common to $\Sigma(x)$ and the appropriate null space.

To complete this discussion we consider singularities of the Riccati matrices in terms of the structure of the solution space $\Sigma(z)$. We noted that the formation of $E(z)$ assumes a basis of type (10) for $\Sigma(z)$. If, however, there is a point z_0 at which $\Sigma(z)$ contains a vector with zeros in positions $n + 1$ to $2n$ then the matrix $V(z)$ in (10) cannot have n linearly independent columns. We might say that the $\mathbf{V}$ solution space has lost rank. In practice a switch is made to the matrix $G(z) = E^{-1}(z)$ before z_0 is reached and the elements of $E^{-1}(z)$ are formed by integrating the

appropriate Riccati equation. The use of $E^{-1}(z)$ assumes a basis for $\Sigma(z)$ of the form

$$\begin{bmatrix} I \\ G(z) \end{bmatrix} U(z), \tag{11}$$

where the columns of U form a basis for the U solution. The structure of $Y(z_0)$ shows that $\det[G(z_0)\,U(z_0)] = 0$, from which it follows that $\det[G(z_0)] = 0$. In short, $\det[E(z)]$ will be singular at any point z where $\Sigma(z)$ and $N([0\ I])$ have a vector $Y(z)$ in common. In terms of the dependent variable $y(z)$ in (7) and (8), $\det[E(z)]$ will be singular at any point z where $\Sigma(z)$ and $N([\gamma_1\ \delta_1])$ have a vector $y(z)$ in common. If $\det[E(z)]$ remains finite throughout some region $\mathcal{g}$ in $\mathbb{R}$ then $N([\gamma_1\ \delta_1])$ does not intersect $\Sigma(z)$ for $z \in \mathcal{g}$. The choice of γ_1 and δ_1 will obviously affect the positions of singularities and the behavior of the Riccati matrix near the singularities.

2.2. *Equations for the Eigenfunction*

The obvious approach to the evaluation of the eigenfunction $Y(z)$ is to integrate the governing equations backwards from $z = x$ to $z = 0$ using $Y(x)$ and $E(x)$ as initial conditions. If, in the format of (1), the linear system in U and V has coefficient matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, then the Riccati equation in E is (see (WB12))

$$E' = \mathcal{B} + \mathcal{A}E + E\mathcal{D} + E\mathcal{C}E. \tag{12}$$

If we eliminate U from the equation $-V' = \mathcal{C}U + \mathcal{D}V$ by means of Eq. (5) we obtain

$$V' = -(\mathcal{C}E + \mathcal{D})V \tag{13}$$

in the vector V. Equations (12) and (13) may be integrated backward from $z = x$ and the upper half of $Y(z)$ is then obtained by means of Eq. (5). If a singularity is approached in the course of the integration then a switch may be made to $G(z)$ and the singularity traversed using the equations

$$-G' = \mathcal{C} + G\mathcal{A} + \mathcal{D}G + G\mathcal{B}G \tag{14}$$

and

$$U' = (\mathcal{A} + \mathcal{B}G)U, \tag{15}$$

where (14) is the analog of (WB6) and (15) is obtained by eliminating V from $U' = \mathcal{A}U + \mathcal{B}V$ by means of

$$V = GU. \tag{16}$$

Computations performed on the Orr–Sommerfeld equation indicated that the differential equations are inherently unstable for backward solution. This difficulty may be readily overcome as shown in Section 3. The reverse integration of an awkward Riccati system may be avoided and the eigenfunction obtained by generating $E(z)$ and $V(z)$ which form the basis (10). The appropriate equations are (12) and (13) with $V(z)$ replaced by $V(z)$ in the latter and with an appropriate set of orthonormal vectors serving as boundary conditions on the columns of $V(z)$ at $z = 0$. The solution

for $Y(z)$ at any station z is given by $[^{E(z)}_f] V(z)\mathbf{c}$, where $\mathbf{c}$ is a suitably normalized solution of

$$[\alpha\ E(x) + \beta]\ V(x)\mathbf{c} = \mathbf{0}. \tag{17}$$

Care must be taken to avoid extreme growth in the elements of the matrix $V(z)$, otherwise $\mathbf{c}$ will not be obtained accurately. At any point where growth has commenced the columns of $V(z)$ may be replaced by an orthonormal set. Continuity of $Y(z)$ at this point will give a relationship between the coefficient vectors $\mathbf{c}$ on either side. If a switch has to be made from $E(z)$ to $G(z)$ at some point z_1, a basis like (11) is initiated at z_1 with $U(z_1)$ suitably chosen. Finally note that the evaluation of $Y(z)$ by forward integration involves the solution of $2n^2$ equations, similar to classical shooting methods, while backward integration involves only $n(n + 1)$ equations.

3. NUMERICAL EXAMPLES

Problem 1.   Consider the problem of computing nontrivial solutions of

$$(d^4y/dz^4) - k^4y = 0 \tag{18}$$

under the boundary conditions

$$y(0) = y'(0) = 0, \qquad y(1) = y'(1) = 0. \tag{19}$$

The eigenvalues are readily given as the positive roots of the transcendental equation $\cos k \cosh k = 1$ and the first of these, which we use in the computation, is $k_1 = 4.7300407$. The eigenfunction $y_j(z)$ associated with the jth eigenvalue k_j and satisfying the normalizing condition $y''(1) = 1$ is

$y_j(z)$

$$= \frac{(\sin k_i - \sinh k_j)(\cos k_jz - \cosh k_jz) - (\cos k_j - \cosh k_j)(\sin k_jz - \sinh k_jz)}{2k_j^2 (\cos k_j \sinh k_j - \sin k_j \cosh k_j)}. \tag{20}$$

This problem with known solution was chosen as a reliable test problem.

The obvious choice of dependent variables for the Riccati approach is $\mathbf{u} = [^{y}_{y'}]$, $\mathbf{v} = [^{y''}_{y'''}]$ and with this choice the problem lies in the class of problems described by Eqs. (1) and (2). The Riccati elements arising from this choice have extremely steep gradients near singularities and it is convenient to use variables $\mathbf{U}$ and $\mathbf{V}$ defined by $\mathbf{U} = \mathbf{u} + \lambda I_1\mathbf{v}$, $\mathbf{V} = \mathbf{v}$, where λ is a free scalar parameter and $I_1 = [^{1}_{0}\ ^{0}_{0}]$. If Eq. (18) is written in the format of (1) with dependent variables $\mathbf{U}$ and $\mathbf{V}$, the coefficient matrices are

$$\mathscr{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = I_2, \mathscr{B} = \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix}, \mathscr{C} = -k^4 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = -k^4 I_3, \mathscr{D} = \begin{bmatrix} 0 & -1 \\ \lambda k^4 & 0 \end{bmatrix}.$$

In the computations k is set equal to k_1 so that $z = x = 1$ is the first characteristic length. If $E(z)$ is introduced through (5), the Riccati equation for $E(z)$ and for its inverse $G(z)$ are given by (12) and (14), respectively. The condition $\mathbf{U}(0) - \lambda I_1 \mathbf{V}(0) = \mathbf{0}$ yields $[E(0) - \lambda I_1] \mathbf{V}(0) = \mathbf{0}$ and this provides the initial condition $E(0) = \lambda I_1$ which permits the initiation of the solution of (12).

System (12) was integrated from $z = 0$ and switches were made between $E(z)$ and $G(z)$ when the respective determinants exceeded unity in modulus. λ was adjusted to reduce slopes at singularities and eventually integrations were performed with $\lambda = -10$. For this value of λ the determinant of $G(1)$ is -0.0790061 so that the final stage of the integration is carried out using Eq. (14). The reverse integration commences at $z = 1$ with Eqs. (14) and (15), so a value of $\mathbf{U}(1)$ is required for initiation. The end condition $[I - \lambda I_1 \, G(1)] \mathbf{U}(1) = \mathbf{0}$ is satisfied by any $\mathbf{U}(1)$ of the form $[^{U_1(1)}_{\ 0}]$. We normalized the first component, $U_1(1)$, to obtain an eigenfunction to match (20) at $z = 1$. The appropriate $\mathbf{U}(1)$ is readily shown to be $\mathbf{U}(1) = [^{\lambda}_{0}]$.
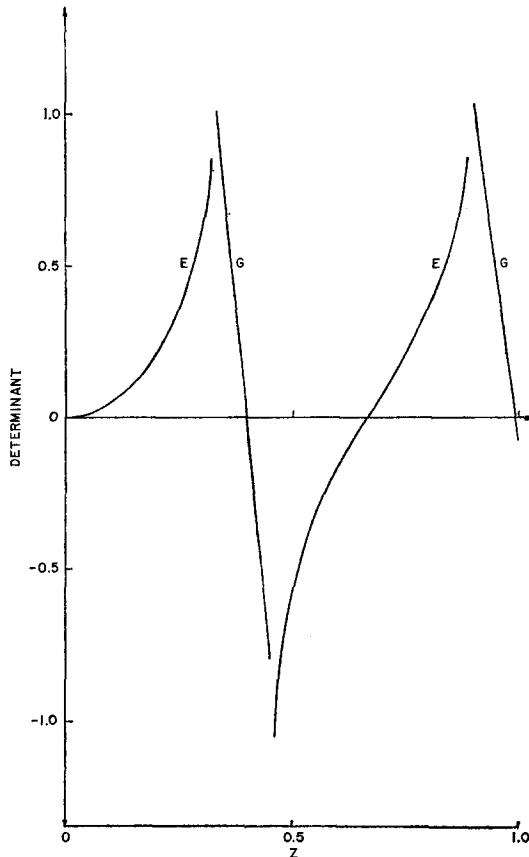

FIG. 1.   Determinants of $E(z)$ and $G(z)$.

Equations (14) and (15), with switches to (12) and (13), were integrated from $z = 1$ to $z = 0$ using a standard fourth-order variable step Runge–Kutta procedure with stepsize control based on local error. Computations were performed on an ICL 1904S computer using single-length arithmetic. The integration passed through three switching points and the first component of $U(z) - \lambda V(z)$ gave a highly accurate approximation to $y_1(z)$ throughout the region $0 \leqslant z \leqslant 1$. Figure 1 shows the profiles of $\det[E(z)]$ and $\det[G(z)]$ and Table I gives computer values of $U_1(z) - \lambda V_1(z)$ at $z = 0.5 \ (0.1) \ 0.9$ which are correct to the number of digits shown.

TABLE I

Eigenfunction $U_1(z) - \lambda V_1(z)$ for Problem 1. There is symmetry about $z = 0.5$

z	0.5	0.6	0.7	0.8	0.9
$10^7 \times (U_1(z) - \lambda V_1(z))$	354920	325265	244935	138422	42261

Problem 2.   Here we consider the computation of the complex solution $\phi(z)$ of the Orr–Sommerfeld equation

$$[(D^2 - \kappa^2)^2 - i\kappa \, \mathrm{Re}\{(\bar{u} - c)(D^2 - \kappa^2) - D^2\bar{u}\}] \, \phi(z) = 0, \tag{21}$$

under the homogeneous boundary conditions

$$\phi(0) = \phi'''(0) = 0, \qquad \phi(1) = \phi'(1) = 0. \tag{22}$$

Here $D \equiv d/dz$, $\bar{u} = 1 - z^2$, $i^2 = -1$, κ and Re are real parameters, and c is a complex parameter. The problem arises in the study of linear instabilities in plane Poiseuille flow. If the Reynolds number, Re, the wave speed, c, and the wave number, κ, have the values 5772.222, 0.264000, 1.020545, respectively, then the system has a nontrivial solution. This is the set of parameter values corresponding to the critical point on the neutral stability curve and the problem of computing accurate values for these parameters has attracted a great deal of attention. The quoted values were obtained by improving results given in a paper by Chock and Schechter [5]. The improvement was achieved by an iterative technique using a Riccati system; the parameter κ was fixed during the iteration and the other parameters were adjusted until the terminating condition (6) was satisfied to the required accuracy.

Introduce the vectors

$$\mathbf{U}(z) = \begin{bmatrix} \phi'(z) \\ \phi''(z) \end{bmatrix}, \qquad \mathbf{V}(z) = \begin{bmatrix} \gamma\phi(z) \\ \delta\phi''(z) \end{bmatrix},$$

where γ and δ are nonzero parameters, and introduce $E(z)$ using Eq. (5). If required, γ and δ permit scaling of the $E(z)$ elements. When Eq. (21) is written in the form of (1) with dependent variables $\mathbf{U}$ and $\mathbf{V}$, the coefficient matrices become

$$\mathscr{A} = \mathscr{D} = 0, \qquad \mathscr{C} = \begin{bmatrix} -\gamma & 0 \\ 0 & -\delta \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} 0 & 1/\delta \\ -\psi/\gamma & (\kappa^2 + \Phi)/\delta \end{bmatrix},$$

where $\Phi = \kappa^2 + i\kappa \, \mathrm{Re}(\bar{u} - c)$ and $\psi = \kappa^2\Phi - 2i\kappa \, \mathrm{Re}$.

System (11) may now be constructed and this is integrated from $z = 0$ using the initial condition $E(0) = 0$. The boundary condition at $z = 1$ may be written as

$$\alpha \mathbf{U}(1) + \beta \mathbf{V}(1) = \mathbf{0} \quad \text{(see Eq. (4)),} \qquad \text{with} \quad \alpha = I_1 \text{ and } \beta = (1/\gamma) I_3,$$

and the resulting terminating condition reduces to $E_2(1) = 0$, where $E_2(z)$ is the component of $E(z)$ in the first row and second column. With the parameters Re, κ, and c as given above the eight scalar equations arising from (12) may be integrated without switching from $z = 0$ to $z = 1$ and the condition $E_2(1) = 0$ is satisfied. To initiate the reverse integration for the eigenfunction we chose $\mathbf{V}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a suitably normalized vector which satisfies $[\alpha\ E(1) + \beta]\ \mathbf{V}(1) = \mathbf{0}$.

In Section 2.2 we stated that the backward integration of (12) suffered from an inherent instability problem. This was overcome by storing $E(z)$ at a set of points $0 < z_1 < z_2 < \cdots < z_m < 1$ during the forward integration. On the reverse sweep the computed value of $E(z_j)$ was replaced by a stored value at z_j when the integration reached this point. This method of successive initiation permitted a straightforward evaluation of the eigenfunction. Computed values of $\phi(z) = V_1(z)/\gamma$ are given in Table II. The results in Table II were obtained using a standard fourth-order Runge–Kutta procedure. The scaling parameters γ and δ were both set equal to 10 and $E(z)$ was stored at $z = 0.1(0.1)0.9$ on the forward sweep. With such a small matrix $E(z)$ no storage problems are encountered in storing at intervals of 0.1; the accuracy given in Table I was also achieved with fewer intermediate storage points.

TABLE II

Real and imaginary parts of Orr–Sommerfeld eigenfunction, $\phi(z)$. Critical point on neutral stability curve with Re = 5772.222, $\kappa = 1.020545$, $c = 0.264000$, $\phi''(1) = 1$

z	0	0.1	0.2	0.3	0.4
$10^5 \times \mathrm{Re}(\phi(z))$	1287	1276	1243	1188	1108
$10^5 \times \mathrm{Im}(\phi(z))$	1148	1138	1109	1060	988
z	0.5	0.6	0.7	0.8	0.9
$10^5 \times \mathrm{Re}(\phi(z))$	1001	863	687	459	204
$10^5 \times \mathrm{Im}(\phi(z))$	893	770	614	408	137

Problem 3. The second problem was repeated using the forward integration method which generates a basis of the form (10). Equations (12) and (13) were integrated from $z = 0$ to $z = 1$, with $V(z)$ replacing $\mathbf{V}(z)$ in Eq. (13). To initiate the integration we have $E(0) = 0$, and $V(0)$ was set equal to the unit 2×2 matrix. Growth in components of $V(z)$ prevents a complete integration from $z = 0$ to $z = 1$ and accordingly a set of points $0 = z_0 < z_1 < z_2 < \cdots < z_m < z_{m+1} = 1$ was selected and at each of these $E(z)$ and $V(z)$ were stored. When $V(z_j)$, $j = 1, 2,..., m$ had been stored $V(z)$ was set equal to the unit 2×2 matrix and the integration from z_j to z_{j+1} was then performed. If $\mathbf{c}_j$, $0 \leqslant j \leqslant m$, is the coefficient vector which applies

in the interval $z_j < z \leqslant z_{j+1}$, then the eigenfunction in the final interval is obtained by finding a nontrivial solution of

$$[\alpha \ E(1) + \beta] \ V(1) \ \mathbf{c}_m = \mathbf{0}. \tag{23}$$

As in Problem 2 we used the normalizing condition $V_2(1) = 1$. These conditions are sufficient for the evaluation of $\mathbf{c}_m$. For $1 \leqslant j \leqslant m$, $\mathbf{c}_{j-1}$ is obtained from $\mathbf{c}_j$ using the recurrence relation

$$V(z_j) \ \mathbf{c}_{j-1} = \mathbf{c}_j, \tag{24}$$

which ensures continuity of $\mathbf{Y}(z)$ at $z = z_j$. The solution $\mathbf{Y}(z)$ at $z = z_j$ is given simply as

$$\mathbf{Y}(z_j) = \begin{bmatrix} E(z_j) \\ I \end{bmatrix} V(z_j) \ \mathbf{c}_{j-1}$$

$$= \begin{bmatrix} E(z_j) \\ I \end{bmatrix} \mathbf{c}_j,$$

where $E(z_j)$ and $V(z_j)$ are the stored values for $z = z_j$.

The eigenfunction was obtained by the method described with successive initiations at intervals of 0.1 and with γ and δ each set equal to 1. The computed values of $V_1(z)$ agreed exactly with the values given in Table II. As in Problem 2 the accuracy given in Table II could be achieved with fewer intermediate starting points.

Considering the abundance of published material on the Orr–Sommerfeld equation there is a surprising paucity of numerical results on eigenfunction evaluation which are suitable for comparison purposes. The first accurate numerical solution was achieved by Thomas [6] using a finite difference method with local truncation error of fourth order in the step length. He employed up to 100 nodes in the interval $0 \leqslant z \leqslant 1$. Thomas considered the unstable mode corresponding to $\kappa = 1$, Re $= 10,000$ and, by extrapolating to zero step length, he obtained the value $0.2375259 + 0.0037404i$ for c. The solution of Eq. (21) is tabulated in [6] for these parametric values and with $\phi(z)$ satisfying the normalizing condition $\phi(0) = 1$. Thomas' parametric values were used here and a numerical approximation to $\phi(z)$ was obtained using the backward and forward integration methods described above

TABLE III

Real and imaginary parts of Orr–Sommerfeld eigenfunction, $\phi(z)$. Unstable mode with Re $= 10,000$, $\kappa = 1$, $c = 0.2375259 + 0.0037404i$, $\phi(0) = 1$

z	0	0.1	0.2	0.3	0.4
$10^5 \times \text{Re}(\phi(z))$	10^5	99187	96725	92550	86542
$10^5 \times \text{Im}(\phi(z))$	0	−6	−26	−59 (1)	−105

z	0.5	0.6	0.7	0.8	0.9
$10^5 \times \text{Re}(\phi(z))$	78519	68204 (1)	55158	38410	16656 (1)
$10^5 \times \text{Im}(\phi(z))$	−167 (1)	−244 (1)	−336 (1)	−403 (1)	−1900

for Problems 2 and 3. The points $z = 0.1(0.1)0.9$ were chosen as intermediate storage points for the backward integration and as initiation points for the forward integration. The computed results, normalized as in [6] are reproduced in Table III. A bracketed 1 following a few entries in Table III indicates that these results differ from Thomas' corresponding results by one unit in the least significant digit.

4. COMMENTS

The Riccati method has certain favorable points relative to traditional shooting methods for the evaluation of eigenvalues of linear differential systems. In the iterative process, for example, each integration sweep involves half the number of equations which are solved using traditional methods. If a shooting method operates by generating a basis of the solution space then steps have to be taken to overcome the effects of growth in the basis components. Keller's [7] method of parallel shooting and the method of orthonormalization described by Conte [8] are both aimed at overcoming the numerical difficulties associated with such problems. An efficient implementation of the orthonormalization procedure has been described in a detailed report by Scott and Watts [9]. In the description of the Orr–Sommerfeld calculations given in the preceding section we noted that the elements of $E(z)$ were well behaved in the forward integration while those of $V(z)$ suffered from a growth problem. The good behavior of the E elements illustrates the point that the Riccati method will overcome some of the numerical difficulties which require special attention when a classical approach is being used. A recent review by Guderley [10] on methods for stiff inhomogeneous two-point boundary value problems describes a projection method for use with this type of problem; the method is closely related to the Riccati approach. This note has shown that if the eigenvalue has been obtained by a Riccati method then an associated eigenfunction may be evaluated with little additional effort.

The use of Riccati transformations for eigenvalues and eigenfunctions of (1) and (3) is readily extended to the case where the system (1) is of order n with ρ boundary conditions at $z = 0$ and s boundary conditions at $z = x$, where $\rho + s = n$ and $\rho > s$ [11]. In this case the matrix R in Eq. (WB3) is of type $\rho \times s$ and the elements of R become infinite if there is a vector common to the solution space and to the null space of a certain $s \times n$ matrix. The singularities may be traversed by switching to a new matrix which also satisfies a $\rho \times s$ matrix Riccati equation. There is a degree of flexibility in the choice of switching points and in the method of implementing the switch. The problem of optimizing the switching process is being investigated. Finally we note that even when R is a square matrix, experience has shown that accuracy may be lost at switching points. The parameter λ was introduced in Problem 1, for example, to ease switching problems.

REFERENCES

1. M. R. Scott, *J. Computational Phys.* **12** (1973), 334.
2. D. M. Sloan and G. Wilks, *J. Inst. Math. Appls.* **18** (1976), 117.
3. M. R. Scott, "Invariant Imbedding and its Applications to Ordinary Differential Equations," Addison–Wesley, Reading, Mass., 1973.
4. G. Wilks and J. S. Bramley, *J. Computational Phys.* (1977).
5. D. P. Chock and R. S. Schechter, *Phys. Fluids* **16** (1973), 329.
6. L. H. Thomas, *Phys. Rev.* **91** (1953), 780.
7. H. B. Keller, "Numerical Methods for Two-Point Boundary Value Problems," Blaisdell, Waltham, Mass., 1968.
8. S. D. Conte, *SIAM Rev.* **8** (1966), 309.
9. M. R. Scott and H. A. Watts, SAND75-0198 (1975), Sandia Laboratories, Albuquerque, New Mexico.
10. K. G. Guderley, *SIAM Rev.* **17** (1975), 416.
11. A. Davey, *J. Computational Phys.* **24** (1977), 331–338.